ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 2, Issue 12 , December 2015

An Arbitrage-Free Market Model for Option Valuation

Akinbo B.J, Ogunrinade S.O, Ojerinde A.T

Department of Mathematics, Federal college of Education, Abeokuta, Nigeria
Department of Mathematics, Federal college of Education, Abeokuta, Nigeria
P.G Student, Departmentof Mathematics, University of Ibadan, Ibadan,Nigeria

ABSTRACT: This paper is an introduction and survey of Black-Scholes Model as an arbitrage-free model which is very useful for Option Valuation. It is a Stochastic processes that represent diffusive dynamics, a common and improved modelling assumption for financial systems.

As the markets are stochastic generally, it becomes very necessary for us to use a more convenient model in order to avoid errors of computations. We include a review of Stochastic Differential equations(SDE), the Itô-lemma which gives a clear picture of Log-normal distribution of a Geometrical Brownian Motion path and solution of Black-Scholes Arbitrage-free model

KEYWORDS: Stochastic Differential Equations,Itô's lemma and arbitrage-free model

I. INTRODUCTION

The standard approach for Option valuation is based on a suitable specification of a stochastic process for the underlying asset. Historically, the protagonist role in describing the evolution of the market prices has been played by the continuous diffusion process. More models for the stochastic dynamics have been proposed market problems, such as Jump-diffusion models and Levy models and so on. We shall focus our attention on the Black-Scholes Model [3].

The free arbitrage opportunity hold a powerful appeal, and provided a foundation for a large finance literature on arbitrage-free models that started with Vasicek(1977) and Cox, Ingersoll and Ross(1985). Their models specify the risk-neutral evolution of the underlying yield-curve as well as the dynamic of the risk premia.
Financial derivative is core area of financial mathematics. In 1972, Black and Scholes tested the result of their model by using the data of Over-the-Counter Market(OTC), they found that result of their Model give tower values than the actual Market values.

Roll and Shastri(1973) found that these differences created due to the imperfect protection of dividend in the OTC market. Black and Scholes (1973) therefore used Itô's lemma mathematical tools which are used to calculate type of stochastic process that also provides helps in the derivation of Black-Scholes formula.
In 1973, Black and Scholes then published their analysis of European Call option in a paper tittled "The pricing of Options and Corporate Liabilities. Robert C. M (1975) that validity of Black-ScholesOption pricing formula depends on the capability of investors to follow a dynamic portfolio strategy in the stock that replicates the payoff structure to the Option. The critical assumption required for such a strategy to be feasible, is that the underlying stock return dynamics can be described by a stochastic process with a continuous sample path. Although in 1996, Duffie and Kan affine that the version of Arbitrage-free models are popular and yield a convenient linear functions of underlying latent factors with parameters that can be calculated from simple system of differential equations. Unfortunately, the canonical affine Arbitrage-free models often exhibit poor empirical time series performance, especially when forecasting future yield, Duffee (2002). (See Jens H. E.Francis. X.D and Glenn.D.R[7]), Jens. H.E et al [7] further Affine Arbitrage-free Class of Nelson- Siegel Term Structure Model where they derived the class affine arbitrage-free dynamic term structure models that approximate the widely-used Nelson- Siegel yield curved specification which can be expressed as slightly restricted version of the canonical representation of the three-factor affine arbitrage-free model. Many Authors have contributed on arbitrage Marketas reviewed in Paul G and XiaoliangZ[17] that a major development in the modeling of interest rates for pricing term structure derivatives is the emergence of models that

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 2, Issue 12 , December 2015

incorporate lognormal volatilities for forward rates while keeping rates stable. It was noted by Heath, Jarrow, and Morton [12] that in the general class of models they developed based on continuously compounded forward rates, lognormal volatilities lead to rates that becomeinfinite in finite time with positive probability. By working instead with various types of discretely compounded rates,Sandmann and Sondermann [18,19], Brace et al. [4], Goldys et al. [8], Miltersen et al. [15], Musiela and Rutkowski [16], and Jamshidian [10,11] have overcome this difficulty and developed well-posed models that indeed admit deterministic diffusion coefficients for the logarithms of forward rates — i.e., lognormal volatilities. The rates themselves are not simultaneously lognormal, but each becomes lognormal under an appropriate change of measure. This class of models — often referred to as *market models* because of their consistency with market conventions — have three principal attractions

II.STOCHASTIC DIFFERENTIAL EQUATIONS

Stochastic Differential Equations (SDEs) are differential equations where stochastic process represents one or more terms and, as a result consequence; the resultant solution will also be stochastic [3]. As more realistic, Mathematical Models become required to take into account random effects and influences in real world systems and SDEs have become essential in the accurate description of such situations [2]. The solutions are continuous-time stochastic processes and methods for the computational solution of stochastic differential equation are based on similar techniques for stochastic dynamic [3].

The aim of this work is to provide a systematic frame work for an understanding of the basic concepts and mathematical tools needed for the development and implementation of Numerical valuation for financial derivative which has become a standard model for financial quantities such as asset prices, interest rate and their derivatives. Unlike deterministic models such as ODEs, which have a unique solution for each approximate initial condition. SDEs have solutions that are continuous-time stochastic process. It is the equation in which one or more of it terms is a stochastic process, thus, resulting in a solution which itself a stochastic process. It can be use to model the randomness of the underlying asset in financial derivative because they give a formal model of how an underlying asset's price changes overtime [1]

Starting with some fundamental concept from calculus that are needed for this work. We consider a general SDE, of the form

$dX_t = \mu(t,x)dt + \sigma(t,x)dW_t X(0) = X_0 \; 0 \leq t \leq T$ (1)

Where $\mu(t,x)$ is the deterministic or drift coefficient and $\sigma(t,x)$ is the diffusion (Noise) where dW_t is the an innovation term representing unpredictable events that occur during the finitesimal interval dt . whileW_t is called a Wiener process [4].

If the diffusion term does not depend on X_t, we say that equation (1) is *additive Noise*,otherwise, the equation has *multiplicative Noise.* The Wiener process, named after Norber is an essential instrument for stochastic process by botanist Brown in 1827, commonly called Brownian Motion.

A Wiener process $W = W_t, \; 0 \leq t \leq T$ is a Gaussian process that depends commonly on time such that

1. $W(0) = 0$(with probability one)
2. For $0 \leq t \leq T$, $E(W(t)) = 0$and for $0 \leq t \leq T$, $Var(W(t) - W(s)) = t - s$
3. For $0 \leq s < t < u < v \leq T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent. The corresponding stochastic integral to (1) above is

$$X_t = X_0 + \int_{t_o}^{t} \mu(s,x)ds + \int_{t_o}^{t} \sigma(s,x)dW_S \qquad (2)$$

Where the last integralS is called Itô integral.

To solve SDEs analytically we introduce the chain's rule for stochastic differential called Itô's lemma [1]

III. ITÔ'S LEMMA

We take into consideration an Itô's lemma by assuming $F(x,t)$ be a twice differentiable function of t and of the random process $X_t$follows the Itô process (1) of the form

$dX_t = \mu_t dt + \sigma_t dW_t \quad t \geq 0$ (3)

then

$$dF_t = \frac{\partial F}{\partial X_t} X_t dt + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2} \sigma_t^2 dt \qquad (4)$$

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 2, Issue 12 , December 2015

Putting (3) into (4) for $X_t dt$ and by using relevant stochastic differential equation, we have

$$dF_t = \left[\frac{\partial F}{\partial S_t}\mu_t + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\sigma_t^2\right]dt + \frac{\partial F}{\partial X_t}\sigma_t dW_t \qquad (5)$$

Supposing the variance $X(t)$ follows a geometric Brownian motion and obeys a stochastic differential equation (1), then the Itô's lemma for any of the function $F(X,t)$ is given as

$$dF(X,t) = \left[\frac{\partial F}{\partial x}\mu X + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 X^2\right]dt + \frac{\partial F}{\partial X}\sigma X dW \qquad (6)$$

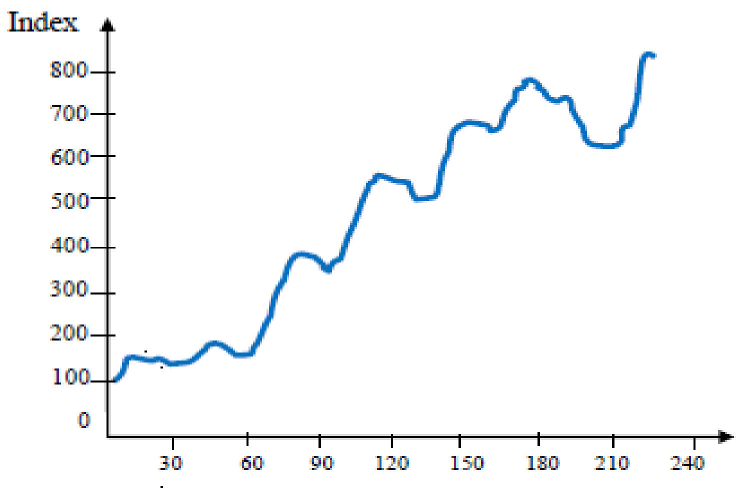
Where μ and σ are constants. We next consider the log-normal distribution, for a stock price process s.Let $F(X,t) = logX$, then

$$\frac{\partial F}{\partial X} = \frac{1}{X}, \quad \frac{\partial F}{\partial t} = 0 \quad and \quad \frac{\partial^2 F}{\partial x^2} = \frac{-1}{X^2} \quad (7)$$

Putting (7) in (6) and by integration, it is trivial that

$$X_T = X_0 Exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma Z\sqrt{T}\right] \qquad (8)$$

Where $Z \sim N(0,1)$. Therefore, it is obvious that stock dynamics follows a log-normal distribution [5], which shows the evolution of the stock price in a Geometric Brownian Motion path using (8) and this graph of simulated data below enhances the understanding of the stochastic behaviours of the underlying assets and assumption that stock returns are log-normally distributed.



Simulation of a geometric Brownian Motion path with $X_o = 100,\ \sigma = 0.20,\ \mu = 0.10,\ and\ N = 300$ as samples drawn from the standard normal distribution

IV. MARKET MODEL

As a model of a financial market, we consider a pair of assets: a *nourisky asset* (bank account) B, and a *risk asset* (stock) which may be represented by their prices *B(t)* and *S(t)* ,$t \in \mathbb{R}_+$. In this case, one speaks of a (B,S)-Market with continuous time. Here, the *risky* component of the (B,S)-Market may be Multidimensional [6]

The assets B and S will be called *underlying assets* or *underlying securities*

We describe the dynamic of the processes as follows

$$dB(t) = rB(t)d$$
$$B(0) = 1$$

and

$$dS(t) = S(t)(\mu dt + \sigma dW(t))$$
$$S(0) = S_0 \qquad (9)$$

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 2, Issue 12 , December 2015

Where r is theinterest rate, μ is the drift parameter and σ is the volatility all assumed to be constant. We get

$$B(t) = e^{rt}$$

and

$$S(t) = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right] \qquad (10)$$

We consider the risky asset S on a filtered probability space$(\Omega, \beta, \mu, \mathbb{F})$, where the filtration $F = \{\beta_t : t \in \mathbb{R}_+\}$ is given by

$$\beta_t = \sigma(W(s): 0 \le s \le t)$$

slightly enlarged to satisfy the usual conditions. Then, the stochastic process S is *adapted* and strictly positives. We called the market model sketched above the *Black-Scholes Model.* We therefore deduced the Black-Scholes formula as follows.

V.BLACK-SCHOLES FORMULA

As an illustrative example of the use of SDE for Option pricing, we consider the European Call(Put) whose value at expiration time T, is $Max\{S(T) - K, 0\}$ $(respectively\ Max\{K - S(T), 0\})$
Where $S(T)$ is the price of the underlying stock, K is the Strike price. The non-arbitrage assumptions of Black-Scholes theory imply that the present values of such an option are

$$C_E(S,T,K) = e^{-rT}\mathbb{E}(Max\{S(T) - K, 0\})$$

and

$$P_E(S,T,K) = e^{-rT}\mathbb{E}(Max\{K - S(T), 0\}) \qquad (11)$$

Where r is the fixed prevailing interest rate during the time interval$[0, T]$, and where the underlying stock price $S(T)$ satisfies the stochastic differential equation (1) of the form

$$dS = rSdt + \sigma S dW_t$$

The value of Call Option can be determined by calculating the expected value (11) explicitly [6], we have

$$C_E(S,T,K) = SN(d_1) - Ke^{-rT}N(d_2)$$

Using Put-Callparity $P_E - C_E = Ke^{-rT} - S$,we have

$$P_E(S,T,K) = Ke^{-rT}N(-d_2) - SN(-d_1)$$

Where

$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma\right)T}{\sigma\sqrt{T}}andd_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma\right)T}{\sigma\sqrt{T}}$

We therefore consider some basic definitions that will aid the completeness of Black-Scholes Model.[5,7]

VI. MARTINGALES

We consider (Ω, β, μ) be a complete probability space and $\mathbb{F}(\mathfrak{B}) = \{\mathfrak{B}_t : t \in [0, \infty)\}$ be a family of sub $\sigma - algebras$ of $\mathfrak{B}$ with the following axions

i. $\mathfrak{B}$. Contains all the $\mu - null$ member of $\mathfrak{B}$;
ii. $\mathfrak{B}_s \subseteq \mathfrak{B}_t$ whenever $t > s \ge 0$,
iii. $\mathbb{F}(\mathfrak{B})$ is right continuous in the sense $\mathfrak{B}_{t_+} = \mathfrak{B}_t, t \ge 0$ where $\mathfrak{B}_{t_+} = \bigcap_{s>t} \mathfrak{B}_s$.

Then, $\mathbb{F}(\mathfrak{B}) = \{\mathfrak{B}_t : t \in [0, \infty)\}$is referred to as *filteration* of $\mathfrak{B}$ and $(\Omega, \beta, \mu, \mathbb{F}(\mathfrak{B}))$ is a *stochastic basis (filtered probability space).*
Definition 1. Assuming $(\Omega, \beta, \mu, \mathbb{F}(\mathfrak{B}))$ for a filtered probability space with $\mathbb{F}(\mathfrak{B}) = \{\mathfrak{B}_t : t \in [0, \infty)\}$ and $\{X(t): t \in [0, \infty)\}$ a stochastic process in (Ω, β, μ) then X is said to be *adapted* to be filtration $\mathbb{F}(\mathfrak{B})$ if $X(t)$ is measurable with respect to $\mathfrak{B}_t$.
Definition 2. We equally consider $\{X(t): t \ge 0\}$ being an adapted $\mathbb{R} - valued$ stochastic process on a filtered probability space $(\Omega, \beta, \mu, \mathbb{F}(\mathfrak{B}))$. Then X is referred to as

i. *Sub martingale* if $X(t)$ is integrable for each $t \geq 0$ and $E(X(t)|\mathfrak{B}_s) \geq X(s)$ *almost surely* whenever $t > s$.

ii. *Super martingale* if $X(t) \in L^{'}(\Omega, \beta, \mu)$ for each $t \geq 0$, $E(X(t)|\mathfrak{B}_s) \geq X(s)$ *almost surely* whenever $t > s$.

iii. *Martingale* if X is both a submartingale and super martingale for $X(t) \in L^{'}(\Omega, \beta, \mu)$ for each $E(X(t)|\mathfrak{B}_s) \geq X(s)$ *almost surely* whenever $t > s$.

Definition. Consider trading within the time horizon $[0, t]$ and the filtered probability space $((\Omega, \beta, \mu, \mathbb{F}(\mathfrak{B}))$. Suppose $\mathfrak{B}_T = \sigma\left(\cup_{t\epsilon[0,T]} \mathfrak{B}_t\right)$. A probability measure μ^* on $(\Omega, \mathfrak{B}_T)$ is an *Equivalent Martingale Measure* if μ^* is equivalent to μ and the discounted price process

$$\tilde{S}(t) = e^{-rt} S(t), \ t\epsilon[0, T].$$

We denote the set of all equivalent martingale measure by $\mathbb{M}(\mu)$ and show that arbitrage opportunities are precluded in a market model if there exists at least one equivalent martingale measure ($\mathbb{M}(\mu) \neq 0$)

VII. ARBITRAGE-FREE MARKET MODEL

We consider $\mathbb{M}(\mu) \neq \emptyset$, and assumed μ^* be a probability measure on (Ω, β) such that $W(t)$ has an $N(mt, t)$ distribution for $W(t) = W(t) - mt$ being a $\mu^* - Brownian\ Motion$ for $0 \leq s < t$,

$$E_{\mu^*}(\tilde{S}(t)|B_s) = E_{\mu^*}(\tilde{S}(t)|\sigma(W_s))$$

$$= E_{\mu^*}(S_0 \exp\left((\lambda - r) - \frac{\sigma^2}{2}\right) t + \sigma W(t))|\sigma(W_s))$$

$$=(S_0 \exp\left((\lambda - r) - \frac{\sigma^2}{2}\right) t) E_{\mu^*}(\exp(\sigma W(t))|\sigma(W_s))$$

$$= (S_0 \exp\left((\lambda - r) - \frac{\sigma^2}{2}\right) t) \exp(\sigma W(s)) E_{\mu^*}(\exp(\sigma(W(t) - W(s)|\sigma(W_s))$$

$$= \left(S_0 \exp\left((\lambda - r) - \frac{\sigma^2}{2}\right) t\right) \exp(\sigma W(s))\, \mathbb{E}_{\mu^*}\left(\exp(\sigma W(t - s))\right).$$

$$= \left(S_0 \exp\left((\lambda - r) - \frac{\sigma^2}{2}\right) t\right) \exp(\sigma W(s)) \exp\left(m(t - s)\sigma + \frac{\sigma^2}{2}(t - s)\right).$$

$$= S_0 \exp\left((\lambda - r)t + m\sigma(t - s) - \frac{\sigma^2 s}{2} + \sigma W(s)\right).$$

Taking $m = \frac{r-\lambda}{\sigma}$, we have

$$E_{\mu^*}(\tilde{S}(t)|B_s) = S_0 \exp\left((\lambda - r) - \frac{\sigma^2}{2}\right) t + \sigma W(s))$$

$$= \tilde{S}(t)$$

Show that the discounted price $\tilde{S}$ is a martingale under $\mu^*(\mu^* \epsilon \mathbb{M}(\mu))$. There is at least one equivalent measure μ^*, showing that the Black-Scholes market model is arbitrage-free.

Under the probability measure μ^* in the last result, we have the dynamic of the process X as

$$dS_t = S_t(\lambda dt + \sigma dW_t)$$

$$= S_t(\lambda dt + \sigma d(\widetilde{W}(t) + mt))$$

$$= S_t\left(\lambda dt + m\sigma dt + \sigma d\widetilde{W}(t)\right)$$

$$= S_t\left(rdt + \sigma d\widetilde{W}(t)\right)$$

with $m = \frac{r-\lambda}{\sigma}$ for which the drift λ is replaced by the risk-free interest rate r.

VIII. SUMMARY AND CONCLUSION

In this paper, we investigated the Black-Scholes model for Option valuation and concentrated our investigation on the fact that the Model is arbitrage-free from option valuation.

In reality, financialmarkets are not frictionless generally.This paper has examined the Black and Scholesmodel as one of the Mathematical tools for option valuation assumed to be simplest model for option pricing.This Model therefore attempts to simplify the markets for both financial assets and derivatives into a set of mathematical rules for a trading strategy of an equivalent martingale measure $\mathbb{M}(\mu)$ that ensured at least one equivalent measure $\mu^{*}\epsilon\mathbb{M}(\mu)$ as claimed.

This model also serves as a basis for a wide range of analysis of markets and reduces the risk of the market transaction.

ACKNOWLEDGMENT: The authors thank Prof. G.O.S Ekhaguerefor passing through his tutelage

REFERENCES

[1] Ames W. (1977) Numerical Method for partial differential equation. Academic press, New York
[2] Avellaneda M. and Laurence P. (2000). Quantitative Modelling of Derivative Securities: From theory of practice. Chapman and Hall.
[3] Black F. and Scholes M. The pricing of options and Corporate Liabilities, Journal ofpolitical Economy 81(3), (1973), 637-654.
[4] Brace, A., Gatarek, D., and Musiela, M.: The market model of interest rate dynamics. Math.Finance 7, 127–155 (1997)
[5] Davis Bundi. N. (2005). Numerical methods for the valuation of financial derivatives
[6] Ekhaguere G.O.S: financing Mathematics, Lecture note
[7] Fouque J., Papanicolaou and Sircar K. (2000). Derivatives in Financial Markets with Stochastic Volatility. Cambridge University Press, Cambridge.
[8] Goldys, B., Musiela, M., Sondermann, D.: Lognormality of rates and term structure models. Working paper, University of New South Wales 1994
[9] Jens H. E., Francis. X.D and Glenn. D.R (2010) Federal Reserve Bank Of San Francisco Working Paper Series
[10] Jamshidian, F.: LIBOR and swap market models and measures. Working paper, Sakura Global Capital, London, 1996
[11] Jamshidian, F.: LIBOR and swap market models and measures. Finance Stochast. 1, 293–330 (1997)
[12] Heath, D., Jarrow, R., Morton, A.: Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. Econometrica 60, 77–105 (1992)
[13] Hull J. (2003). Options, Futures and other Derivatives. Pearson Education Inc, Fifth Edition, New Jersey.
[14] Kolb R. (1999). Futures, Options and Swaps. Blackwell Publishers Ltd, Third Edition,UK.
[15Miltersen, K.R., Sandmann, K., Sondermann, D.: Closed-form solutions for term structure derivatives with lognormal interest rates. J. Finance 70, 409–430 (1997)
[16] Musiela, M., Rutkowski, M.: Martingale methods in financial modeling. Berlin Heidelberg New York: Springer 1997
[17] Paul G and Xiaoliang Z (2000). Arbitrage-free discretization of lognormal forward Libor and swap rate models, Finance Stochast. 4, 35–68 (2000)
[18] Sandmann, K., Sondermann, D.: A term structure model and the pricing of interest rate options. Rev. Futures Markets 12, 391–423 (1993)
[19] Sandmann, K., Sondermann, D.: Log-normal interest rate models: stability and methodology. Discussion paper B-398, Department of Statistics, University of Bonn 1997