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Coefficient Bounds for a New Subclasses of m-Fold Symmetric Bi-Univalent Functions

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ABSTRACT:In the present investigation, we consider two new subclasses $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk U. For functions belonging to the two classes introduced here, we derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Several related classes are also considered and connections to earlier known results are made.

KEY WORDS: Analytic function, Bi-univalent function, m-Fold symmetric function, m-Fold symmetric bi-univalent function.

I.INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ } and \text{ } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all unctions in $\mathcal{A}$ which are univalent in U.

It is well know that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \tfrac{1}{4}\right),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. We denote by Σ the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1). Lewin [12] investigated the class of bi-univalent functions Σ and obtained a bound $|a_2| \leq 1.51$. Motivated by the work of Lewin [12], Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. Some examples of bi-univalent functions are $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log(\frac{1+z}{1-z})$ (see also Srivastava et al. [18]). The coefficient estimate problem involving the bound of $|a_n|(n \in \mathbb{N} \setminus \{1,2\})$ for each $f \in \Sigma$ is still an open problem [18].

For each function $f \in \mathcal{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)}\left(z \in U;\ m \in \mathbb{N}\right) \tag{3}$$

is univalent and maps the unit disk U into a region with m-fold symmetric. A function is said to be m-fold symmetric (see [11,16]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}\left(z \in U;\ m \in \mathbb{N}\right). \tag{4}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (4). The functions in the class $\mathcal{S}$ are said to be one-fold symmetric. Each bi-univalent function generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (4) and the series expansion for f^{-1}, which was recently proven by Srivastava et al. [19], is given as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1} + \cdots, \tag{5}$$

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where $f^{-1}=g$, we denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m=1$, formula (5) coincide with formula (2). Some examples of m-fold symmetric bi-univalent functions are given as follows [19]:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, [-\log(1-z^m)]^{\frac{1}{m}} \text{ and } \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}}, \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}} \ and \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}}, respectively.$$

Brannan and Taha [6] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses$\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ of starlike and convex function of order $\beta(0\le\beta<1)$ respectively (see [15]).The classes $\mathcal{S}_\Sigma^*(\alpha)$and $\mathcal{K}_\Sigma(\alpha)$of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}_\Sigma^*(\alpha)$ and$\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [18] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of bi-univalent functions ([2], [3], [4], [8], [9], [13], [17], [20]).

The aim of this paper is to introduce two new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses. Several related classes are also considered and connections to earlier known results are made.

In order to derive our main results, we have to recall here the following lemma [7].

Lemma 1. If $h\in\mathcal{P}$, then $|b_k|\le 2$ for each $k\in\mathbb{N}$, where $\mathcal{P}$ is the family of all functions h, analytic in U, for which

$$R\big(h(z)\big)>0 \qquad (z\in U),$$

where

$$h(z)=1+b_1z+b_2z^2+\cdots \qquad (z\in U).$$

II. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\alpha)$

Definition 1. A function $f(z)\in\Sigma_m$ given by (4) is said to be in the class $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\alpha)$ $(0<\alpha\le1;\ 0\le\mu\le\lambda\le1)$ if the following conditions are satisfied:

$$\left|\arg\left(\frac{\lambda\mu z^3 f'''(z)+(2\lambda\mu+\lambda-\mu)z^2f''(z)+zf'(z)}{\lambda\mu z^2f''(z)+(\lambda-\mu)zf'(z)+(1-\lambda+\mu)f(z)}\right)\right|<\frac{\alpha\pi}{2}(z\in U) \tag{6}$$

and

$$\left|\arg\left(\frac{\lambda\mu w^3 g'''(w)+(2\lambda\mu+\lambda-\mu)w^2g''(w)+wg'(w)}{\lambda\mu w^2g''(w)+(\lambda-\mu)wg'(w)+(1-\lambda+\mu)g(w)}\right)\right|<\frac{\alpha\pi}{2}(w\in U), \tag{7}$$

where the function $g=f^{-1}$ is given by (5).

Theorem 1. Let $f(z)\in\mathcal{M}_{\Sigma_m}(\lambda,\mu;\alpha)$ $(0<\alpha\le1;\ 0\le\mu\le\lambda\le1)$ be of form (4). Then

$$|a_{m+1}|\le\frac{2\alpha}{\sqrt{2\alpha m(m+1)(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)+m(m-m\alpha-2\alpha)(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2}} \tag{8}$$

and

$$|a_{2m+1}|\le\frac{2(m+1)\alpha^2}{m^2(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2}+\frac{\alpha}{m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)}. \tag{9}$$

proof. It follows from (6) and (7) that

$$\frac{\lambda\mu z^3 f'''(z)+(2\lambda\mu+\lambda-\mu)z^2f''(z)+zf'(z)}{\lambda\mu z^2f''(z)+(\lambda-\mu)zf'(z)+(1-\lambda+\mu)f(z)}=[p(z)]^\alpha \tag{10}$$

and

$$\frac{\lambda\mu w^3 g'''(w)+(2\lambda\mu+\lambda-\mu)w^2g''(w)+wg'(w)}{\lambda\mu w^2g''(w)+(\lambda-\mu)wg'(w)+(1-\lambda+\mu)g(w)}=[q(w)]^\alpha, \tag{11}$$

where the functions $p(z)$ and $q(w)$are in $\mathcal{P}$ and have the following series representations:

$$p(z)=1+p_mz^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \tag{12}$$

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and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{13}$$

Now, equating the coefficients in (10) and (11), we obtain

$$m(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)a_{m+1} = \alpha p_m, \tag{14}$$

$$2m(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)a_{2m+1} - m(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2 a_{m+1}^2 = \alpha p_{2m} + \frac{1}{2}\alpha(\alpha - 1)p_m^2 \tag{15}$$

$$-m(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)a_{m+1} = \alpha q_m \tag{16}$$

and

$$2m(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)[(m+1)a_{m+1}^2 - a_{2m+1}] - m(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2 a_{m+1}^2 = \alpha q_{2m} + \frac{1}{2}\alpha(\alpha - 1)q_m^2. \tag{17}$$

From (14) and (16), we find

$$p_m = -q_m \tag{18}$$

and

$$2m^2(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2 a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \tag{19}$$

From (15), (17) and (19), we get

$$2\alpha m(m+1)(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)a_{m+1}^2 + m(m - m\alpha - 2\alpha)(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2 a_{m+1}^2 = \alpha^2(p_{2m} + q_{2m}). \tag{20}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m} + q_{2m})}{2\alpha m(m+1)(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu) + m(m - m\alpha - 2\alpha)(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2}. \tag{21}$$

Applying Lemma (1) for the coefficients p_{2m} and q_{2m}, we have

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{2\alpha m(m+1)(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu) + m(m - m\alpha - 2\alpha)(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2}}.$$

This gives the desired bound for $|a_{m+1}|$ as asserted in (8).
Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (17) from (15), we get

$$4m(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)a_{2m+1} - 2m(m+1)(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)a_{m+1}^2 = \alpha(p_{2m} - q_{2m}) + \frac{1}{2}\alpha(\alpha - 1)(p_m^2 - q_m^2). \tag{22}$$

It follows from (18), (19) and (22) that

$$a_{2m+1} = \frac{(m+1)\alpha^2(p_m^2 + q_m^2)}{4m^2(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2} + \frac{\alpha(p_{2m} - q_{2m})}{4m(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)}.$$

Apply Lemma (1) once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we readily obtain

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{m^2(1 + m\lambda - m\mu + m\lambda\mu + m^2\lambda\mu)^2} + \frac{\alpha}{m(1 + 2m\lambda - 2m\mu + 2m\lambda\mu + 4m^2\lambda\mu)}.$$

This completes the proof of Theorem (1).

III. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \beta)$

Definition 2. A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \beta)$ $(0 \leq \beta < 1;\ 0 \leq \mu \leq \lambda \leq 1)$ if the following conditions are satisfied:

$$\mathrm{Re}\left(\frac{\lambda\mu z^3 f'''(z) + (2\lambda\mu + \lambda - \mu)z^2 f''(z) + zf'(z)}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)}\right) > \beta \qquad (z \in U) \tag{25}$$

and

$$\mathrm{Re}\left(\frac{\lambda\mu w^3 g'''(w) + (2\lambda\mu + \lambda - \mu)w^2 g''(w) + wg'(w)}{\lambda\mu w^2 g''(w) + (\lambda - \mu)wg'(w) + (1 - \lambda + \mu)g(w)}\right) > \beta \qquad (w \in U), \tag{26}$$

where the function $g = f^{-1}$ is given by (5).

Theorem 2. Let $f(z) \in \mathcal{M}_{\Sigma_m}(\lambda, \mu; \beta)$ $(0 \leq \beta < 1;\ 0 \leq \mu \leq \lambda \leq 1)$ be of form (4). Then

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$$|a_{m+1}| \le \sqrt{\frac{2(1-\beta)}{m(m+1)(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)-m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2}}. \qquad (27)$$

and

$$|a_{2m+1}| \le \frac{2(m+1)(1-\beta)^2}{m^2(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2} + \frac{(1-\beta)}{m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)}. \qquad (28)$$

Proof. It follows from (25) and (26) that there exist $p, q \in \mathcal{P}$ such that

$$\frac{\lambda\mu z^3 f'''(z) + (2\lambda\mu+\lambda-\mu)z^2 f''(z) + zf'(z)}{\lambda\mu z^2 f''(z) + (\lambda-\mu)zf'(z) + (1-\lambda+\mu)f(z)} = \beta + (1-\beta)p(z) \qquad (29)$$

and

$$\frac{\lambda\mu w^3 g'''(w) + (2\lambda\mu+\lambda-\mu)w^2 g''(w) + wg'(w)}{\lambda\mu w^2 g''(w) + (\lambda-\mu)wg'(w) + (1-\lambda+\mu)g(w)} = \beta + (1-\beta)q(w), \qquad (30)$$

where $p(z)$ and $q(w)$ have form (12) and (13), respectively.
It follows from (29) and (30) that

$$m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)a_{m+1} = (1-\beta)p_m, \qquad (31)$$

$$2m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)a_{2m+1} - m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2 a_{m+1}^2 = (1-\beta)p_{2m}, \qquad (32)$$

$$-m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)a_{m+1} = (1-\beta)q_m \qquad (33)$$

and

$$2m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)[(m+1)a_{m+1}^2 - a_{2m+1}] - m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2 a_{m+1}^2 = (1-\beta)q_{2m}. \qquad (34)$$

From (31) and (33), we find

$$p_m = -q_m \qquad (35)$$

and

$$2m^2(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2 a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \qquad (36)$$

Adding (32) and (34), we have

$$[2m(m+1)(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu) - 2m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \qquad (37)$$

Therefore, we obtain

$$a_{m+1}^2 = \frac{(1-\beta)(p_{2m}+q_{2m})}{2m(m+1)(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu) - 2m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2}. \qquad (38)$$

Applying Lemma (1) for coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \le \sqrt{\frac{2(1-\beta)}{m(m+1)(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)-m(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2}}.$$

This is the bound on $|a_{m+1}|$ asserted in (25).
Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (34) from (32), we get

$$4m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)a_{2m+1} - 2m(m+1)(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)a_{m+1}^2 = (1-\beta)(q_{2m}-p_{2m})$$

or, equivalently,

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{(1-\beta)(q_{2m}-p_{2m})}{4m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)}.$$

Then, in view of (36) and applying Lemma (1) for the coefficients p_m, p_{2m}, q_m and q_{2m}, we have

$$|a_{2m+1}| \le \frac{2(m+1)(1-\beta)^2}{m^2(1+m\lambda-m\mu+m\lambda\mu+m^2\lambda\mu)^2} + \frac{(1-\beta)}{m(1+2m\lambda-2m\mu+2m\lambda\mu+4m^2\lambda\mu)}.$$

This completes the proof of Theorem (2).

If we set $\mu = 0$ in Theorem (1) and Theorem (2), then the classes $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\beta)$ reduce to the classes $\mathcal{M}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda;\beta)$ and thus, we obtain the following corollaries:

Corollary 1. Let $f(z)$ given by (4) be in the class $\mathcal{M}_{\Sigma_m}(\lambda;\alpha)$ $(0 < \alpha \le 1;\ 0 \le \lambda \le 1)$. Then

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$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{2\alpha m(m+1)(1+2m\lambda)+m(m-m\alpha-2\alpha)(1+m\lambda)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{m^2(1+m\lambda)^2}+\frac{\alpha}{m(1+2m\lambda)}.$$

Corollary2. Let f(z) given by (4) be in the class $\mathcal{M}_{\Sigma_m}(\lambda;\beta)$ $(0 \leq \beta < 1;\ 0 \leq \lambda \leq 1)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{m(m+1)(1+2m\lambda)-m(1+m\lambda)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2(1+m\lambda)^2}+\frac{(1-\beta)}{m(1+2m\lambda)}.$$

The classes $\mathcal{M}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda;\beta)$ are respectively defined as follows:

Definition 3. A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{M}_{\Sigma_m}(\lambda;\alpha)$ $(0 < \alpha \leq 1;\ 0 \leq \lambda \leq 1)$ if the following conditions are satisfied:

$$\left|\arg\left(\frac{\lambda z^2 f''(z)+zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right)\right| < \frac{\alpha\pi}{2}\,(z \in U)$$

and

$$\left|\arg\left(\frac{\lambda w^2 g''(w)+wg'(w)}{\lambda wg'(w)+(1-\lambda)g(w)}\right)\right| < \frac{\alpha\pi}{2}\,(w \in U),$$

where the function $g = f^{-1}$ is given by (5).

Definition 4. A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{M}_{\Sigma_m}(\lambda;\beta)$ $(0 \leq \beta < 1;\ 0 \leq \lambda \leq 1)$ if the following conditions are satisfied:

$$\mathrm{Re}\left(\frac{\lambda z^2 f''(z)+zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right) > \beta \qquad (z \in U)$$

and

$$\mathrm{Re}\left(\frac{\lambda w^2 g''(w)+wg'(w)}{\lambda wg'(w)+(1-\lambda)g(w)}\right) > \beta \qquad (w \in U),$$

where the function $g = f^{-1}$ is given by (5).
If we set $\mu = 0$ and $\lambda = 1$ in Theorem (1) and Theorem (2), then the classes $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda,\mu;\beta)$ reduce to the classes $\mathcal{K}_{\Sigma_m}^{\alpha}$ and $\mathcal{K}_{\Sigma_m}^{\beta}$ and thus, we obtain the following corollaries:

Corollary 3. Let f(z) given by (4) be in the class $\mathcal{K}_{\Sigma_m}^{\alpha}$ $(0 < \alpha \leq 1)$. Then

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{2\alpha m(m+1)(1+2m)+m(m-m\alpha-2\alpha)(1+m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{m^2(1+m)^2}+\frac{\alpha}{m(1+2m)}.$$

Corollary4. Let f(z) given by (4) be in the class $\mathcal{K}_{\Sigma_m}^{\beta}$ $(0 \leq \beta < 1)$. Then

$$|a_{m+1}| \leq \sqrt{\frac{2(1-\beta)}{m(m+1)(1+2m)-m(1+m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2(1+m)^2}+\frac{(1-\beta)}{m(1+2m)}.$$

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The classes $\mathcal{K}_{\Sigma_m}^{\alpha}$ and $\mathcal{K}_{\Sigma_m}^{\beta}$ are defined in following way:

Definition 5. A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{K}_{\Sigma_m}^{\alpha}$ $(0 < \alpha \leq 1)$ if the following conditions are satisfied:

$$\left|\arg\left(\frac{zf''(z)}{f'(z)} + 1\right)\right| < \frac{\alpha\pi}{2} (z \in U)$$

and

$$\left|\arg\left(\frac{wg''(w)}{g'(w)} + 1\right)\right| < \frac{\alpha\pi}{2} (w \in U),$$

where the function $g = f^{-1}$ is given by (5).

Definition 6. A function $f(z) \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{K}_{\Sigma_m}^{\beta}$ $(0 \leq \beta < 1)$ if the following conditions are satisfied:

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \beta \qquad (z \in U)$$

and

$$\mathrm{Re}\left(\frac{wg''(w)}{g'(w)} + 1\right) > \beta \qquad (w \in U),$$

where the function $g = f^{-1}$ is given by (5).

If we set $\mu = 0$ and $\lambda = 0$ in Theorem (1) and Theorem (2), then the classes $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \beta)$ reduce to the classes $\mathcal{S}_{\Sigma_m}^{\alpha}$ and $\mathcal{S}_{\Sigma_m}^{\beta}$ and thus, we obtain the following corollaries, which were proven earlier by Altinkaya and Yalçin [1].

Corollary 5. Let f(z) given by (4) be in the class $\mathcal{S}_{\Sigma_m}^{\alpha}$ $(0 < \alpha \leq 1)$. Then

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha + 1}}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{m^2} + \frac{\alpha}{m}.$$

Corollary6. Let f(z) given by (4) be in the class $\mathcal{S}_{\Sigma_m}^{\beta}$ $(0 \leq \beta < 1)$. Then

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{(1-\beta)}{m}.$$

For one-fold symmetric bi-univalent functions, Theorem (1) and Theorem (2) lead us to Corollary (7) and Corollary (8), respectively, which were proven by Keerthi and Raja [10], where the classes $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \alpha)$ and $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \beta)$ reduce to the classes $\mathcal{M}_{\Sigma}(\lambda, \mu; \alpha) \equiv B_{\Sigma}(\alpha, \lambda, \mu)$ and $\mathcal{M}_{\Sigma_m}(\lambda, \mu; \beta) \equiv \mathcal{N}_m(\beta, \lambda, \mu)$, respectively.

Corollary 7. Let f(z) given by (1) be in the class $B_{\Sigma}(\alpha, \lambda, \mu)$ $(0 < \alpha \leq 1; 0 \leq \mu \leq \lambda \leq 1)$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1 + 2\lambda - 2\mu + 6\lambda\mu) + (1 - 3\alpha)(1 + \lambda - \mu + 2\lambda\mu)^2}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1 + \lambda - \mu + 2\lambda\mu)^2} + \frac{\alpha}{(1 + 2\lambda - 2\mu + 6\lambda\mu)}.$$

Corollary 8. Let f(z) given by (1) be in the class $\mathcal{N}_{\Sigma}(\beta, \lambda, \mu)$ $(0 \leq \beta < 1; 0 \leq \mu \leq \lambda \leq 1)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(2+4\lambda-4\mu+12\lambda\mu)-(1+\lambda-\mu+2\lambda\mu)^2}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\lambda-\mu+2\lambda\mu)^2} + \frac{(1-\beta)}{(1+2\lambda-2\mu+6\lambda\mu)}.$$

Remark. For the case of one-fold symmetric bi-univalent functions,

- Putting $\mu = 0$ in Theorem (1) and Theorem (2), we obtain the corresponding results given by Keerthi and Raja [10].
- Putting $\mu = 0$ and $\lambda = 0$ in Theorem (1) and Theorem (2), we obtain the corresponding results given by Li and Wang [13].
- Putting $\mu = 0$ and $\lambda = 1$ in Theorem (1) and Theorem (2), we obtain the corresponding results given by Murugusundaramoorthy at el. [14].

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