ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

Vol. 7, Issue 11 , November 2020

Fundamental Solutions of Axisymmetric Boundary Problems

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ABSTRACT: Under the Hamilton system, the dual form governing equations of the cylindrical problem are established using displacements and stresses as the basic variables, and the problem is accordingly transformed into finding the eigen values and eigen solutions by adopting the method of separation of variables. Furthermore, solutions for zero eigen values and non-zero eigen values are systematic studied.

KEY WORDS: Hamilton System, Governing Equation, Eigenvalue.

I. INTRODUCTION

Saint Venant's principle is a basic principle of elasticity, which was put forward by Saint Venant in 1855 [1]. Its main content is: the stress in the object caused by external loads distributed in a small area of the elastomer is only related to the resultant force and moment at a distance near the load, and the distribution of the load only affects the stress distribution near its action area. Many scholars have studied the correction of Saint Venant's principle and found that it holds in most practical problems.

In the boundary value problem of elastic mechanics, strictly speaking, the boundary conditions given by external force and displacement should be satisfied strictly, but it is very difficult to give the solution that fully satisfies the boundary conditions mathematically [2, 3].Therefore, boundary conditions can be put forward by some equivalent forms in the process of solving elastic problems. This kind of equivalence will bring some approximation in mathematics, but it is found that the error brought by this approximation is local in long-term practice [4, 5].

In this paper, under Hamilton system, the fundamental governing equations of three-dimensional problems are established, and the complete solution space and all Saint Venant solutions problems are obtained.

II. SOLUTION METHOD

For an elastic cylinder, the governing equation for the nonzero eigensolution is:

$$(\mathbf{H}-\mu\mathbf{I})\psi=0 \tag{1}$$

The dimensionless geometric equation in cylindrical coordinates is expressed as

$$\begin{gathered}\varepsilon_r=\frac{\partial u}{\partial r},\quad \varepsilon_\theta=\frac{1}{r}\left(\frac{\partial v}{\partial\theta}+u\right),\quad \varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}-\frac{v}{r}\right),\\ \varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right),\quad \varepsilon_{\theta z}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z}\right),\quad \varepsilon_z=\frac{\partial w}{\partial z}.\end{gathered} \tag{2}$$

For the convenience of discussion, we express the displacement and stress components as follows

$$\bar{\mathbf{q}}=\{\bar{u},\bar{v},\bar{w}\}^T \tag{3}$$

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

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$$\bar{\mathbf{p}}=\frac{\partial L}{\partial \dot{\mathbf{q}}}=\left\{\begin{array}{c} r\,G^{*}\left(\dot{\bar{u}}+\dfrac{\partial \bar{w}}{\partial r}\right) \\ G^{*}\left(\dfrac{\partial \bar{w}}{\partial \theta}+r\dot{\bar{v}}\right) \\ \lambda^{*}\left(r\dfrac{\partial \bar{u}}{\partial r}+\bar{u}+\dfrac{\partial \bar{v}}{\partial \theta}+\dot{\bar{w}}\right)+2G^{*}\dot{\bar{w}} \end{array}\right\}=\left\{\begin{array}{c} r\,\bar{\tau}_{rz} \\ r\,\bar{\tau}_{\theta z} \\ r\,\bar{\sigma}_{z} \end{array}\right\} \tag{4}$$

According to the theory of elasticity, the solution can be expressed by the complete Papkovitch Neuber type:

$$\begin{aligned} p_1&=\mu G r\left[B_r-2a_2\frac{\partial}{\partial r}\left(B_0+rB_r\right)\right]\\ p_2&=\mu G\left[rB_\theta-2a_2\frac{\partial}{\partial \theta}\left(B_0+rB_r\right)\right]\\ p_3&=a_1 G r\left(\frac{\partial B_r}{\partial r}+B_r+\frac{\partial B_\theta}{\partial \theta}\right)-2a_2\mu^2 rG\left(B_0+rB_r\right)\\ u&=B_r-a_2\frac{\partial}{\partial r}\left(B_0+rB_r\right)\\ v&=B_\theta-a_2\frac{\partial}{\partial \theta}\left(\frac{B_0}{r}+B_r\right)\\ w&=-\mu a_2\frac{\partial}{\partial r}\left(B_0+rB_r\right) \end{aligned} \tag{5}$$

in which

$$\left\{B_0,\quad B_r,\quad B_\theta\right\}=\sum_n\left\{R_0^{(n)}(r),\quad R_r^{(n)}(r),\quad iR_\theta^{(n)}(r)\right\}e^{in\theta}e^{\mu z} \tag{6}$$

Substituting Eq. (5) into Eq. (4), we get

$$\begin{aligned} &\left(\frac{d^2}{dr^2}+\frac{d}{dr}+\mu^2-\frac{n^2}{r^2}\right)R_0^{(n)}=0\\ &\left(\frac{d^2}{dr^2}+\frac{d}{dr}+\mu^2-\frac{n^2+1}{r^2}\right)R_r^{(n)}+\frac{2n}{r^2}R_\theta^{(n)}=0\\ &\left(\frac{d^2}{dr^2}+\frac{d}{dr}+\mu^2-\frac{n^2+1}{r^2}\right)R_\theta^{(n)}+\frac{2n}{r^2}R_r^{(n)}=0 \end{aligned} \tag{7}$$

The lateral condition can be expressed by primal variable and dual variable

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International Journal of Advanced Research in Science, Engineering and Technology
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$$\begin{aligned}
& l\left[ra_4\frac{\partial\overline{u}}{\partial r}++a_1\overline{p}_3\right]+mG^*\left(r\frac{\partial\overline{v}}{\partial r}+\frac{\partial\overline{u}}{\partial\theta}-\overline{v}\right)=0 \\
& lG^*\left(r\frac{\partial\overline{v}}{\partial r}+\frac{\partial\overline{u}}{\partial\theta}-\overline{v}\right)+m\left[+2a_1G^*r\frac{\partial\overline{u}}{\partial r}+a_1\overline{p}_3\right]=0 \\
& l\left[2a_1G^*\left(\frac{\partial\overline{v}}{\partial\theta}+\overline{u}\right)+a_1\overline{p}_3\right]+mG^*a_4\left(\frac{\partial\overline{v}}{\partial\theta}+\overline{u}\right)=0
\end{aligned} \tag{8}$$

When the eigenvalue is zero, there are two groups of fundamental solutions. They are

$$\begin{cases}
\overline{\psi}_1^{(0)}=\{\sin\theta,\quad\cos\theta,\quad 0,\quad 0,\quad 0,\quad 0\}^T \\
\overline{\psi}_2^{(0)}=\{-\cos\theta,\quad\sin\theta\quad 0,\quad 0,\quad 0,\quad 0\}^T \\
\overline{\psi}_3^{(0)}=\{0,\quad 0,\quad 1,\quad 0,\quad 0,\quad 0\}^T \\
\overline{\psi}_4^{(0)}=\{0,\quad 1,\quad 0,\quad 0,\quad 0,\quad 0\}^T
\end{cases} \tag{9}$$

and

$$\begin{cases}
\overline{\psi}_1^{(1)}=\{0,\quad 0,\quad -r\sin\theta,\quad 0,\quad 0,\quad 0\}^T \\
\overline{\psi}_2^{(1)}=\{0,\quad 0,\quad r\cos\theta,\quad 0,\quad 0,\quad 0\}^T \\
\overline{\psi}_3^{(1)}=\{-v^*r,\quad 0,\quad 0,\quad 0,\quad 0,\quad E^*r\}^T \\
\overline{\psi}_4^{(1)}=\{0,\quad 0,\quad \varphi,\quad G^*r\partial_r\varphi,\quad G^*\left(\partial_\theta\varphi+r^2\right),\quad 0\}^T
\end{cases} \tag{10}$$

For the case of nonzero eigenvalues, the general solution can be described by Bessel functions:

$$\begin{cases}
R_0^{(n)}=c_1^{(n)}J_n(\mu r) \\
R_r^{(n)}=c_2^{(n)}J_{n+1}(\mu r)+c_3^{(n)}J_{n-1}(\mu r) \\
R_\theta^{(n)}=-c_2^{(n)}J_{n+1}(\mu r)+c_3^{(n)}J_{n-1}(\mu r)
\end{cases} \tag{11}$$

Using the method of separating variables, we have

$$\psi=\sum_n\psi^{(n)}e^{in\theta}e^{\mu z} \tag{12}$$

The solution is

ISSN: 2350-0328

International Journal of Advanced Research in Science, Engineering and Technology

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$$u^{(n)} = J_{n+1}c_2^{(n)} + J_{n-1}c_3^{(n)} - a_2 \frac{d}{dr}\left(rJ_n c_1^{(n)} + rJ_{n+1}c_2^{(n)} + rJ_{n-1}c_3^{(n)}\right)$$

$$v^{(n)} = -\frac{n}{r}a_2 J_n c_1^{(n)} - (1+na_2)J_{n+1}c_2^{(n)} + (1-na_2)J_{n-1}c_3^{(n)}$$

$$w^{(n)} = -\mu a_2\left(J_n c_1^{(n)} + rJ_{n+1}c_2^{(n)} + rJ_{n-1}c_3^{(n)}\right) \tag{13}$$

$$p_1^{(n)} = 2G\mu a_2\left[(\mu rJ_{n-1} + nJ_n)c_1^{(n)} + (n+a_3)rJ_{n+1}c_2^{(n)} - (n-a_3)rJ_{n-1}c_3^{(n)}\right]$$

$$p_2^{(n)} = -2G\mu a_2 i\left[nJ_n c_1^{(n)} + (n+a_3)rJ_{n+1}c_2^{(n)} + (n-a_3)rJ_{n-1}c_3^{(n)}\right]$$

$$p_3^{(n)} = \mu a_2 E\left[-\mu rJ_n c_1^{(n)} + (a_3-2)rJ_n c_2^{(n)} - (a_3-2)rJ_n c_3^{(n)}\right]$$

Substituting the lateral condition, we get

$$2\mu a_7 J_1 c_1^{(0)} + (J_1 - 2\mu a_7 J_0)\left(c_2^{(0)} - c_3^{(0)}\right) = 0$$

$$(\mu J_0 - 2J_1)\left(c_2^{(0)} + c_3^{(0)}\right) = 0 \tag{14}$$

$$2\mu a_7(\mu J_0 - J_1)c_1^{(0)} + \left[(1+2a_7)\mu J_0 + 2\left(\mu^2 a_7 - 1\right)J_1\right]\left(c_2^{(0)} - c_3^{(0)}\right) = 0$$

III. NUMERICAL EXAMPLE

Fig.1 and Fig.2 are displacement component and stress component of the first five non-zero eigenvalues results obtained by numerical calculation, respectively. The figures indicate that non-zero eigenvalues are necessary to describe local effects caused by boundary conditions.

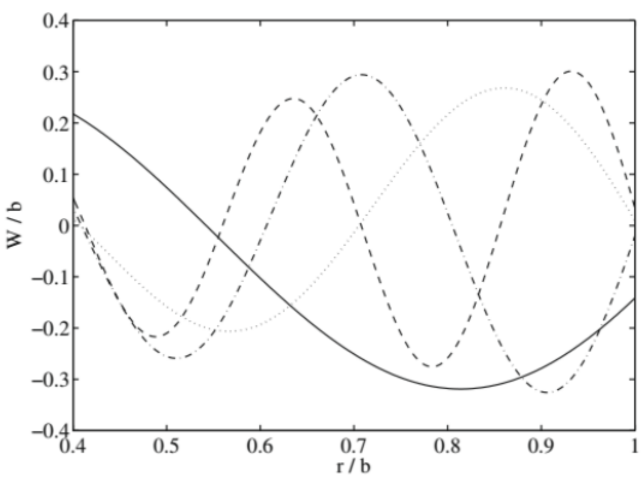

Fig1. Displacement component of the first five non-zero eigenvalues

ISSN: 2350-0328
International Journal of Advanced Research in Science, Engineering and Technology
Vol. 7, Issue 11 , November 2020

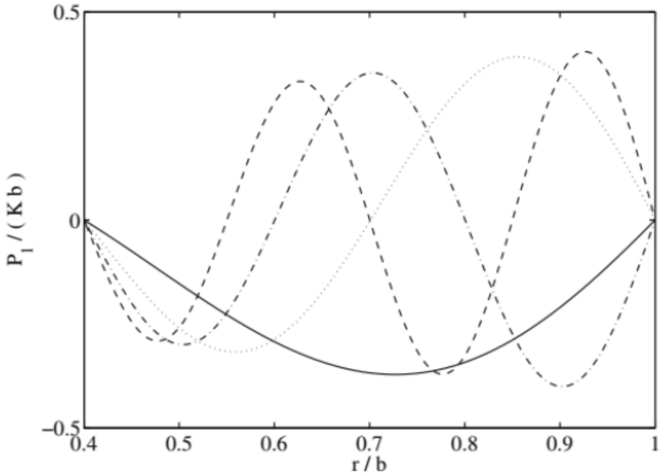

Fig 2.Stress component of the first five non-zero eigenvalues

IV. CONCLUSION AND FUTURE WORK

The Hamiltonian system is applied to find analytical solutions of Saint-Venant problems of elastic cylinders. The study shows that non-zero eigensolutions include the torsion and bending groups charactered by local deformations, while zero eigensolutions are composed of all the overall deformation solutions such as the traditional tension and bending problems. The study shows that zero eigensolutions are composed of all the overall deformation solutions such as the traditional tension and bending problems, while non-zero eigensolutions include the torsion and bending groups charactered by local deformations.

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