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Aspects of univalent functions with negative coefficients and Some Geometric Properties with Hadamard Product

Azhar hassan Hussein, Maha hadi Abed sajet , Eman ali reda ,Doaa hadi khudhair

Department of Science of Statistics, College of Administration and Economics, University of Al-Qadisiya, Diwaniya, Iraq

ABSTRACT: In our paper we study a class $L(\alpha,\beta,b,0,\mu)$,which consists of analytic and univalent functions with negative coefficients in the open unit disk U={z∈C:|z|<1}defined by Hadamard product (or convolution) with Operator, we obtain coefficient bounds and extreme points for this class. Also distortion theorem using fractional calculus techniques and some results for this class are obtained.
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KEY WORDS AND PHRASES: Univalent Function, Fractional Calculus, Hadamard Product, Distortion The orem .Operator,,Extreme Point..

Definition (1.1): We say that the function f of complex variable is analytic in a domain D if is differentiable at every point in that domain D.

Definition (1.2): A function f analytic in a domain D is said to be univalent there if it does not take the same value twice that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D.
In other word, f is one-to-one (or injective) mapping of D onto another domain. If f(z) assumes the same value more than one, then f is said to be multivalent (p-valent) in D. Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, n \in \mathbb{N} = \{1,2,3,\dots\} \qquad (1.1)$$

Which are analytic and univalent in the unit disk $U = \{z \in \mathbf{C} : |z| < \}$. If a function f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, n \in \mathbb{N} = \{1,2,3,\dots\} \qquad (1.2)$$

is in the class A, the convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, z \in U \qquad (1.3)$$

Let S denote the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \qquad (1.4)$$

Definition (1.3)[4]: A function f ∈ A is said to be starlike function of order α if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (0 \leq \alpha < 1; z \in U) \qquad (1.5)$$

We denote the class of all starlike functions of order α in U by $S^*(\alpha)$.

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Definition (1.4)[4]: A function $f \in A$ is said to be convex function of order α if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,(0\le\alpha<1;z\in U) \qquad (1.6)$$

We denote the class of all convex functions of order α in U by $C(\alpha)$.
Note that $S^*(0) = S^*$, $C(0) = C$ and $C \subset S^* \subset A$, and the Koebe function is starlike but not convex, where the Koebe function given by. $K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$
is the most famous function in the class A , which maps U onto **C** minus a slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$.

Definition (1.5) [4]: A function f analytic in the unit disk U is said to be close-to-convex function of order α ($0 \le \alpha < 1$) if there is a convex function g such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>\alpha, \quad \text{for all} \quad z\in U \qquad (1.7)$$

We denote by $K(\alpha)$ the class of close-to-convex functions of order α ,f is normalized by the usual conditions $f(0)=f'(0)-1=0$,These functions are connected by the relation $C \subset S^* \subset K$.

Definition (1.6) [7]: The fractional integral of order $\delta(0<\delta)$ is defined by

$$D_z^{-\delta}f(z) = \frac{1}{\Gamma(\delta)}\int_0^z \frac{f(t)}{(z-t)^{1-\delta}}dt, \qquad (1.8)$$

Where f(z) is an analytic function in a simply connected region of Z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition (1.7)[7]: The fractional derivative of order δ is defined by

$$D_z^{\delta}f(z) = \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\delta}}dt, \qquad (1.9)$$

Where f(z) is as in Definition (1.6) and the multiplicity of $(z-t)^{-\delta}$ is removed like Definition (1.6).

Definition (1.8)[7]:[Under the Condition of Definition(1.7).
The fractional derivative of order $n+\delta$ ($n = 0,1,2,\dots$) is defined by

$$D_z^{n+\delta}f(z)=\frac{d^n}{dz^n}D_z^{\delta}f(z)$$

From definition (1.6) and (1.7) by applying a simple calculation, we get

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(2+\delta)}z^{1+\delta}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1+\delta)}a_n z^{n+\delta}, \qquad (1.10)$$

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(2-\delta)}z^{1-\delta}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\delta)}a_n z^{n-\delta}. \qquad (1.11)$$

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zxDefinition(1.9)[4]: Let X be a topological vector space over the field of **C** and let E be a subset of X .A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1 - t)z, 0 < t < 1$, as a proper convex combination of two distinct points y and z in E.

Definition(1.10)[4]: Radius of starlikeness of a function f is the largest r_0 , $0 < r_0 < 1$ for which it is starlike in $|z| < r_0$.

Definition(1.11)[4]: Radius of convexity of a function f is the largest r_1, $0 < r_1 < 1$ for which it is convex in $|z| < r_1$. .

1.1 Basic Lemmas and Theorems

Theorem (1.2.1): If $\alpha \geq 0$, $0 \leq \beta < 1$ and $\gamma \in R$, then $\operatorname{Re} w > \alpha|w - 1| + \beta$ if and only if $\operatorname{Re}(w(1 + \alpha e^{i\gamma}) - \alpha e^{i\gamma}) > \beta$, where w be any complex number.

Theorem (1.2.2): With the same condition as in Theorem (1.2.1), $\operatorname{Re} w > \alpha$ if and only if $|w - (1 + \alpha)| < |w + (1 - \alpha)|$. These can be found in [2].
The next theorem is due to Alexander`s Theorem [2].

Theorem (1.2.3): If f be an analytic function in U with $f(0) = f'(0) - 1 = 0$, then $f \in C$ if and only if $zf' \in S^*$.

Theorem (1.2.4)(Distortion Theorem[4]): For each $f \in A$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, |z| = r < 1$$

For each $z \in U, z \neq 0$, equ ality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem (1.2.5)(Growth Theorem[4]): For each $f \in A$

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r < 1$$

For each $z \in U, z \neq 0$,equality occurs if and only if f is a suitable rotation of the Koebe function.

Lemma(1.2.6)(Schwarz Lemma): Let f be analytic in the unit disk U with $f(0) = 0$ and $|f(z)| < 1$ in U .Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in U .Strict inequality holds in both estimates unless f is a rotation of the disk $f(z) = e^{i\theta} z$.

The integral operator of $f \in S$ for $\lambda > -1, \mu \geq 0$ is denoted by L_λ^μ and defined as following:

$$L_\lambda^\mu f(z) = \frac{(\lambda+1)^\mu}{\Gamma(\mu)} \int_0^1 t^\lambda \left(\log \frac{1}{t}\right)^{\mu-1} \frac{f(zt)}{t} dt \qquad = z - \sum_{n=2}^{\infty} \left(\frac{1}{n}\right)^\mu a_n z^n (\lambda = 0 \mu \geq 0, f \in S)$$

(2.1)

The operator defined by (2.1) is known as the Komatu operator[5].

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A function $f \in S$, $z \in U$ is said to be in the claas $L(\alpha,\beta,b,0,\mu)$ if and only if it satisfies the inequality

$$\mathrm{Re}\left\{\beta\frac{L_\lambda^\mu f(z)}{z}+(1-\beta)(L_\lambda^\mu f(z))'+\alpha z(L_\lambda^\mu f(z))''\right\}>1-|b| \tag{2.2}$$

For some $\alpha(\alpha\geq 0), -1\leq\beta\leq 0, b\in\mathbb{C}, \lambda=0$ and $\mu \geq 0$, for all $z\in U$.
The class $L(\alpha, 0, 1-\gamma, \lambda, 0)$ was introduced b Altintas[1] who obtained several results concerning this class .The class $L(\alpha,\beta,b,0,\mu)$ was introduced by Srivastava and Owa[7].
The class $L(\alpha,\beta,b,0,\mu)$ was introduced by Atshan and Kulkarni[3].

2.2 Main Results

In the following theorem, we derive the coefficient inequality for the class $L(\alpha,\beta,b,0,\mu)$.

Theorem (2.2.1): Let $f \in S$ .Then f is in the class $L(\alpha,\beta,b,0,\mu)$ if and only if

$$\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{1}{n}\right)^{\mu}a_n\leq|b| \tag{2.3}$$

The result (2.3) is sharp.

Proof: Assume that $f\in L(\alpha,\beta,b,0,\mu)$ .Then, we find from (2.2) that

$$\mathrm{Re}\left\{\beta\left[1-\sum_{n=2}^{\infty}a_n\left(\frac{1}{n}\right)^{\mu}z^{n-1}\right]+(1-\beta)\left[1-\sum_{n=2}^{\infty}na_n\left(\frac{1}{n}\right)^{\mu}z^{n-1}\right]\right.$$
$$\left.+\alpha z\left[-\sum_{n=2}^{\infty}n(n-1)a_n\left(\frac{1}{n}\right)^{\mu}z^{n-2}\right]\right\}>1-|b|.$$

If we choose z to be the real and let $z\to 1$, we get

$$1-\sum_{n=2}^{\infty}[\beta+n(1-\beta+\alpha n-\alpha)]\left(\frac{1}{n}\right)^{\mu}a_n\geq 1-|b|,$$

Which is equivalent to (2.3).conversely, assume that (2.3) is true. Then, we have

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$$\left|\beta\frac{L_{\lambda}^{\mu}f(z)}{z}-(1-\beta)(L_{\lambda}^{\mu}f(z))'-\alpha z(L_{\lambda}^{\mu}f(z))''-1\right| \le \sum_{n=2}^{\infty}\left[\beta+n(1-\beta+\alpha n-\alpha)\right] \backslash$$

$$\left(\frac{1}{n}\right)^{\mu} a_n \le |b|.$$

This implies that $f \in L(\alpha,\beta,b,0,\mu)$. The result (2.3) is sharp for the function

$$f(z)=z-\frac{|b|}{\left[\beta+n(1-\beta+\alpha n-\alpha)\right]\left(\frac{1}{n}\right)^{\mu}}z^{n}, n\ge 2 \qquad (2.4)$$

In the following theorem, we obtain interesting properties of the class $f \in L(\alpha,\beta,b,0,\mu)$.

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